Twin-Width 2: Small Classes

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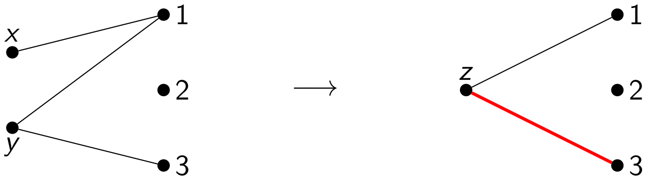
Contraction

Contracting vertices x, y into z:

1. if both xu, yu are edges, then zu is an edge
2. if neither xu, yu are edges, then zu is not an edge
3. if exactly one of xu, yu are edges, then zu is an error edge



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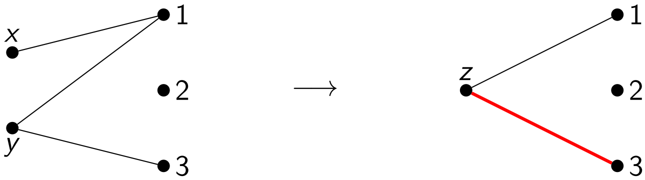
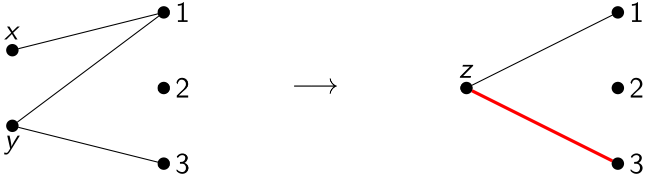


Trigraph: between two vertices either no edge, or a normal edge, or an error edge.

Contraction

Contracting vertices x, y into z:

1. if xu, yu have the same state (no edge, normal edge, or red edge), then zu is given that state
2. otherwise zu is an error edge



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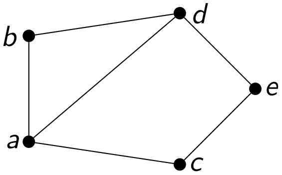
Contraction Sequence

Sequence of trigraphs which:

- Starts with G
- Ends with the 1-vertex graph
- Progresses by contracting two vertices at each step



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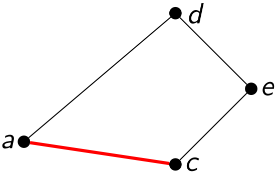

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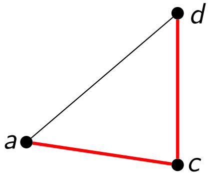

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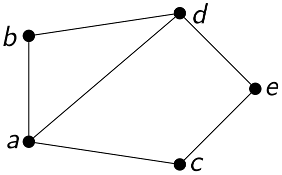

A contraction sequence is a t-sequence if the red degree never exceeds t.
Twin-width $tww(G)$ is the smallest t such that G admits a t-sequence.

Important Results

- Twin-width is closed by induced subgraph.
- Twin-width is invariant by complementation.
- Twin-width 'generalises' clique-width: $tww(G) = O(cw(G))$

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Theorem (Twin-Width I, 2020)

Given G with n-vertices, a t-sequence for G, and a FO formula ϕ, one can test $G \models \phi$ in $O(f(t, \phi) \cdot n)$.

Small Classes

Definition

A class of graphs is *small* if it has at most $n!c^n$ graphs on the vertex set $\{1, \ldots, n\}$.

Examples: trees, bounded tree-width.

Theorem (Norine et al., 2006)

Any class of graphs which excludes a minor is small.

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Generalisation:

Theorem

Any class of graphs with bounded twin-width is small.

Application

Lemma (Bender & Canfield, 1978)

Cubic graphs are not a small class.

Neglecting $O(1)^n$, there are $n^{3n/2}$ cubic graphs.
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Cubic graphs have unbounded twin-width.

No explicit construction known!

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Similar proof by counting for interval graphs, unit disk graphs.

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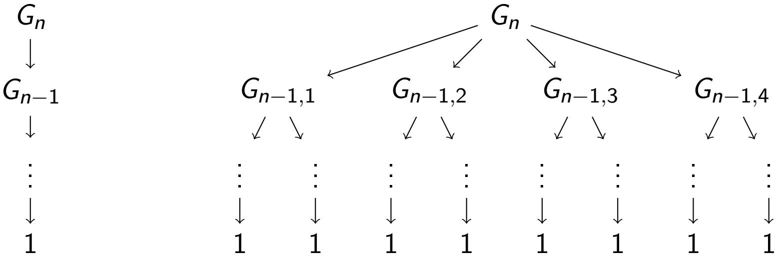
Open: (small, twin-width unknown)

- classes with polynomial expansion
- finite subgraphs of a Cayley graph

Versatile Trees of Contractions

G with n vertices has a p-versatile tree of t-contractions if

- G has red degree at most t
- There exist $x_1, y_1, \ldots, x_{np}, y_{np}$ distinct vertices such that each G/x_iy_i has a p-versatile tree of t-contractions.



Example

- Paths have a 1-sequence: pick one extremity, contract the last edge.
- Paths do not have a p-versatile tree of 1-contractions for any $p > 0$.
- Paths have a $1/3$-versatile tree of 2-contractions: contract any edge.

Versatile Twin-Width Theorem

Theorem

For any t, there exist t' and $p > 0$ such that if $tww(G) \leq t$, then G has a p-versatile tree of t'-contractions.

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For any t, there exist t' and $p > 0$ such that if $tww(G) \leq t$, then G has a p-versatile tree of t'-contractions.

Applications:

- Main lemma in the small classes theorem.
- Finding parallelized contraction sequences of length $\log n$. Application: $O(\log n)$-adjacency labelling scheme.
- Approximation algorithm for DOMINATING SET (not yet published)

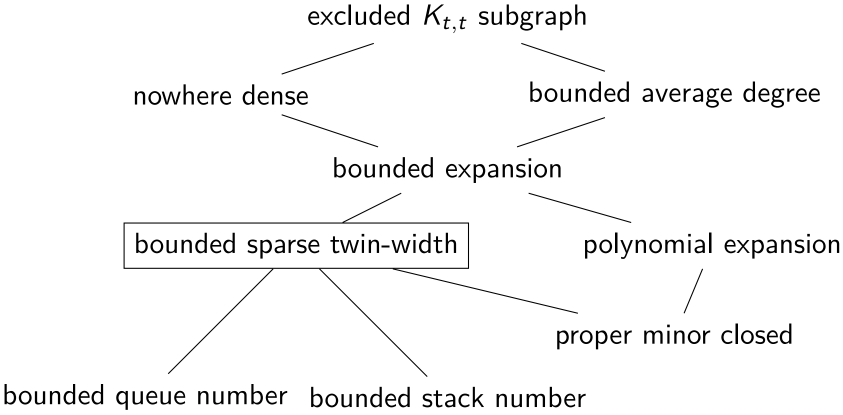
Sparse Twin-Width Theorem

Theorem

Let $\mathcal{C}$ be a class of bounded twin-width, closed by induced subgraph. TFAE:

1. *$\mathcal{C}$ has bounded expansion*
2. *$\mathcal{C}$ is nowhere dense*
3. *$\mathcal{C}$ has bounded average degree*
4. *$\mathcal{C}$ excludes $K_{t,t}$ as subgraph for some t*
5. *the subgraph closure of $\mathcal{C}$ has bounded twin-width*
6. *$\mathcal{C}$ is d-grid free for some d*

Bounded Sparse Twin-Width



Summary

Versatile twin-width: a very useful tool

- Bounded twin-width implies small
- Many other applications

Sparse twin-width is well behaved:

- A ‘unique’ notion of sparsity for bounded twin-width
- Includes proper minor closed, bounded stack/queue number

Open questions:

- Small classes conjecture
  Special cases: polynomial expansion, Cayley graphs
- Constructing cubic graphs with unbounded twin-width
- Characterization of bounded sparse twin-width?