# Twin-width of tournaments

Colin Geniet    Stéphan Thomassé

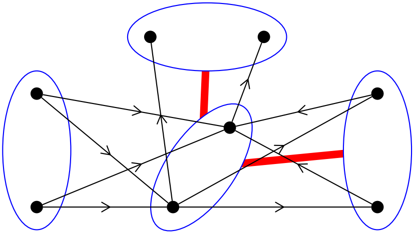
ENS Lyon

ESA 2023

# Error graph (directed graphs)

Let $G$ a directed graph, $P$ a partition of $V(G)$

*Error graph $G/P$:*



Two parts $X, Y \in P$ can be:

- *homogeneous*:
  if there is an edge $X \to Y$, then there must be *all* edges $X \to Y$ (and vice versa)
- otherwise, error

# Twin-width (directed graphs)

Contraction sequence:

- Start with the partition of $V(G)$ into singletons.
- Merge two parts iteratively, until only one part is left.

$\rightarrow$ sequence of error graphs.

# Twin-width (directed graphs)

Contraction sequence:

- Start with the partition of $V(G)$ into singletons.
- Merge two parts iteratively, until only one part is left.

$\rightarrow$ sequence of error graphs.

Twin-width: $\mathrm{tww}(G) \leq k$ iff there is a contraction sequence where error graphs have degree $\leq k$.

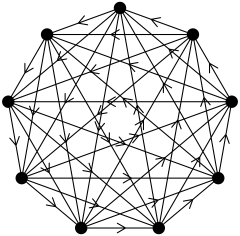
# Twin-width (directed graphs)

Contraction sequence:

- Start with the partition of $V(G)$ into singletons.
- Merge two parts iteratively, until only one part is left.

$\rightarrow$ sequence of error graphs.

Twin-width: $\mathrm{tww}(G) \leq k$ iff there is a contraction sequence where error graphs have degree $\leq k$.



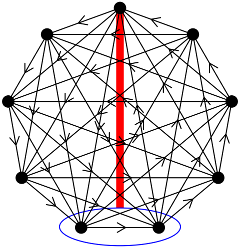
# Twin-width (directed graphs)

Contraction sequence:

- Start with the partition of $V(G)$ into singletons.
- Merge two parts iteratively, until only one part is left.

$\rightarrow$ sequence of error graphs.

Twin-width: $\text{tww}(G) \leq k$ iff there is a contraction sequence where error graphs have degree $\leq k$.



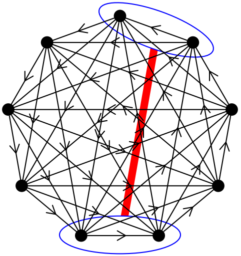
# Twin-width (directed graphs)

Contraction sequence:

- Start with the partition of $V(G)$ into singletons.
- Merge two parts iteratively, until only one part is left.

$\rightarrow$ sequence of error graphs.

Twin-width: $\mathrm{tww}(G) \leq k$ iff there is a contraction sequence where error graphs have degree $\leq k$.



# Dynamic programming with twin-width

**Theorem (Bonnet, Kim, Thomassé, Watrigant, '20)**

*Given a graph $G$, a FO formula $\phi$, and a contraction sequence of width $t$, one can test if $G$ satisfies $\phi$ in time $f(\phi, t) \cdot n$.*

Example: dominating set of size $k$

$$\exists x_1, \ldots, x_k . \forall y. \bigvee_{i=1}^{k} x_i = y \vee (x_i \to y)$$

# Dynamic programming with twin-width

**Theorem (Bonnet, Kim, Thomassé, Watrigant, '20)**

*Given a graph $G$, a FO formula $\phi$, and a contraction sequence of width $t$, one can test if $G$ satisfies $\phi$ in time $f(\phi, t) \cdot n$.*

Example: dominating set of size $k$

$$\exists x_1, \ldots, x_k . \forall y. \bigvee_{i=1}^{k} x_i = y \vee (x_i \to y)$$

**Problem**

*Is there a FPT algorithm to compute twin-width?*

# Dynamic programming with twin-width

**Theorem (Bonnet, Kim, Thomassé, Watrigant, '20)**

*Given a graph $G$, a FO formula $\phi$, and a contraction sequence of width $t$, one can test if $G$ satisfies $\phi$ in time $f(\phi, t) \cdot n$.*

Example: dominating set of size $k$

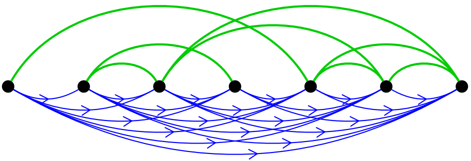
$$\exists x_1, \ldots, x_k . \forall y . \bigvee_{i=1}^{k} x_i = y \vee (x_i \to y)$$

**Problem**

*Is there a FPT algorithm to ~~compute~~ approximate twin-width (up to any function)?*

# Ordered graphs

Ordered graph $G = (V, E, <)$



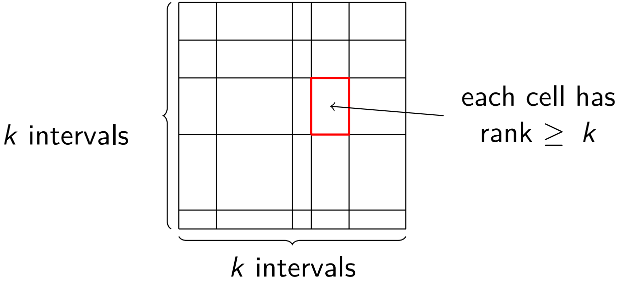
# Twin-width of ordered graphs

**Theorem (BGOSTT '21)**

*For ordered graphs (graph + total order on vertices):*

- *there is a FPT approximation of twin-width,*
- *bounded twin-width* $\iff$ *no* $k$*-rank minor for some* $k$
- *and many more characterisations...*

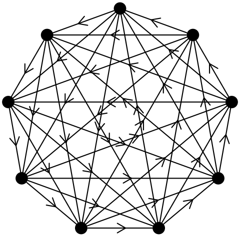
$k$-rank minor in $(G, E, <)$: in the adjacency matrix of $E$ ordered by $<$



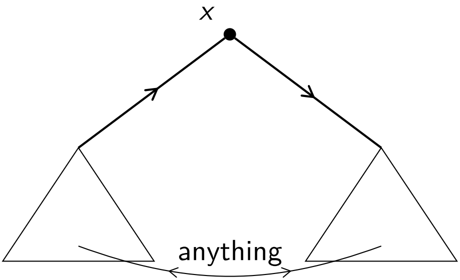
# Tournaments

Goal: adapt the results on ordered graphs to more general structures.

Tournament: clique with a choice of orientation of each edge.



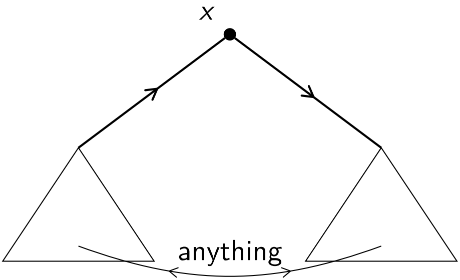
# Binary search trees




If $y$ is a left (resp. right) descendant of $x$,
then $y \to x$ (resp. $x \to y$).

*BST order:*
left-to-right order on a BST.

# Binary search trees




If $y$ is a left (resp. right) descendant of $x$,
then $y \to x$ (resp. $x \to y$).

*BST order:*
left-to-right order on a BST.

**Lemma (G., T.)**

*There is a function $f$ such that if $T$ is a tournament, $<$ a BST order, then*

$$\mathrm{tww}(T) \leq \mathrm{tww}(T, <) \leq f(\mathrm{tww}(T))$$

# Main results

**Lemma (G., T.)**

*There is a function $f$ such that if $T$ is a tournament, $<$ a BST order, then*

$$\operatorname{tww}(T) \leq \operatorname{tww}(T, <) \leq f(\operatorname{tww}(T))$$

**Corollary**

*There is a FPT approximation of twin-width on tournaments.*

# Main results

**Lemma (G., T.)**

*There is a function $f$ such that if $T$ is a tournament, $<$ a BST order, then*

$$\mathrm{tww}(T) \leq \mathrm{tww}(T,<) \leq f(\mathrm{tww}(T))$$

**Corollary**

*There is a FPT approximation of twin-width on tournaments.*

Remark:
the result also applies to structures $(V, E, T)$ where $(V, T)$ is a tournament.

Tournaments take the role of linear orders.

# Conclusion

- Previously known:
  twin-width is well-behaved on ordered graphs [BGOSTT '21]

# Conclusion

- Previously known:
  twin-width is well-behaved on ordered graphs [BGOSTT '21]
- Twin-width is also well-behaved on tournaments
  (and anything overlayed on a tournament).

# Conclusion

- Previously known:
  twin-width is well-behaved on ordered graphs [BGOSTT '21]
- Twin-width is also well-behaved on tournaments
  (and anything overlayed on a tournament).
- Other classes of graphs in which twin-width is approximable?
  (Unit segment graphs?)

# Conclusion

- Previously known:
  twin-width is well-behaved on ordered graphs [BGOSTT '21]
- Twin-width is also well-behaved on tournaments
  (and anything overlayed on a tournament).
- Other classes of graphs in which twin-width is approximable?
  (Unit segment graphs?)
- Approximation of twin-width in general?

# Conclusion

- Previously known:
  twin-width is well-behaved on ordered graphs [BGOSTT '21]
- Twin-width is also well-behaved on tournaments
  (and anything overlayed on a tournament).
- Other classes of graphs in which twin-width is approximable?
  (Unit segment graphs?)
- Approximation of twin-width in general?
- Other uses of BST in tournaments?
  (Ailon, Charikar, Newman '08: approx of feedback arc set)

# Conclusion

- Previously known:
  twin-width is well-behaved on ordered graphs [BGOSTT '21]
- Twin-width is also well-behaved on tournaments
  (and anything overlayed on a tournament).
- Other classes of graphs in which twin-width is approximable?
  (Unit segment graphs?)
- Approximation of twin-width in general?
- Other uses of BST in tournaments?
  (Ailon, Charikar, Newman '08: approx of feedback arc set)

# Conclusion

- Previously known:
  twin-width is well-behaved on ordered graphs [BGOSTT '21]
- Twin-width is also well-behaved on tournaments
  (and anything overlayed on a tournament).
- Other classes of graphs in which twin-width is approximable?
  (Unit segment graphs?)
- Approximation of twin-width in general?
- Other uses of BST in tournaments?
  (Ailon, Charikar, Newman '08: approx of feedback arc set)

Questions?