Graph parameters and Groups

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based on work with

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32 KIAS Combinatorics Workshop

Groups

A group Γ has:

- an associative product $a \cdot b$
- a neutral element 1_Γ
- inverses a^{-1}

Groups

A group Γ has: but sometime it is written with addition!

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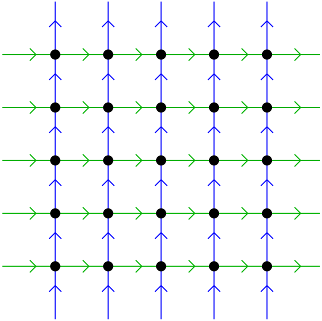
My groups usually are:

- non-commutative,
- infinite,
- but with a finite generating set.

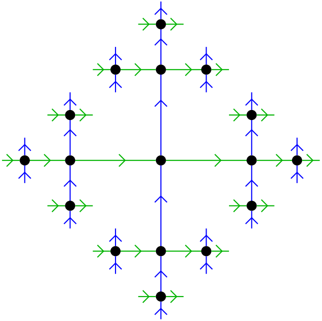
Going back to graphs

For Γ a group, $S \subset \Gamma$ a finite generating set.
Cayley graph $\mathrm{Cay}(\Gamma, S)$:

- vertices $V = \Gamma$,
- edges $E = \{(x, x \cdot s) \mid x \in \Gamma, s \in S\}$.



$\Gamma = \mathbb{Z}^2$, $S = \{(1,0), (0,1)\}$



$\Gamma = \mathbb{F}(\{a,b\})$, $S = \{a, b\}$

Tree-width and groups

Graph G with finite tree-width:
partition $\mathcal{P}$ of $V(G)$ where

- $G/\mathcal{P}$ is a tree
- parts of $\mathcal{P}$ have bounded size

only correct for bounded degree!

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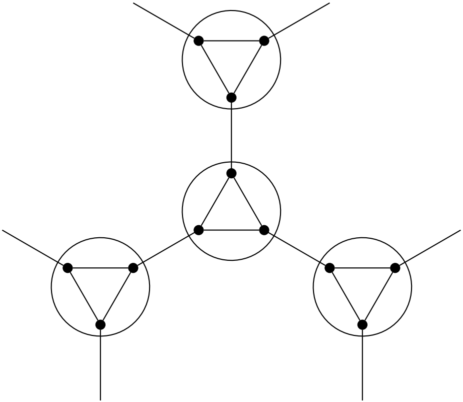

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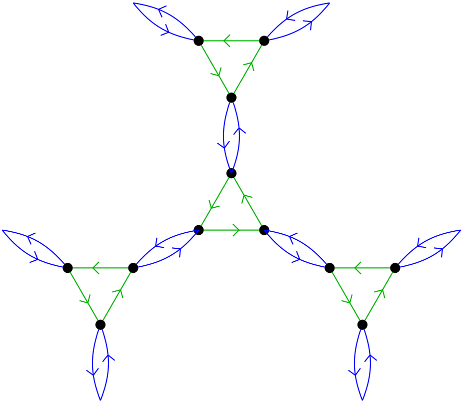

Generators a, b
satisfying $a^3 = 1,\ b^2 = 1$

(This is called $\mathrm{PSL}_2(\mathbb{Z})$)

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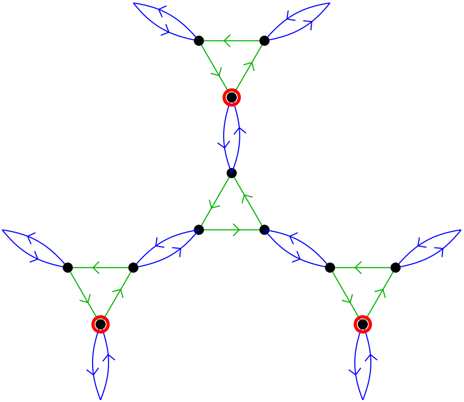
Group Γ virtually free:
subgroup Λ where

- Λ is free
- $[\Gamma : \Lambda] := |\Gamma/\Lambda|$ is finite



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Λ is freely generated by
$baba^{-1}$, $ba^{-1}ba$
$[\Gamma : \Lambda] = 6$

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Theorem (Kuske & Lohrey, '05)

For any group Γ and finite generating set S, $Cay(\Gamma, S)$ has finite tree-width if and only if Γ is virtually free.

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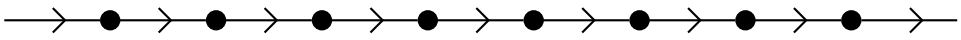
Remark

In particular: for any finite generating sets S, S', $\text{Cay}(\Gamma, S)$ has finite tree-width if and only if $\text{Cay}(\Gamma, S')$ does.

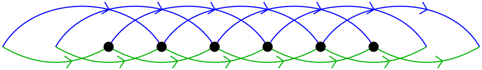
So many Cayley graphs...

A single group can have several Cayley graphs.

$\mathbb{Z}$ with the obvious generator $\{1\}$:



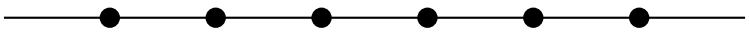
$\mathbb{Z}$ with stupid generators $\{2, 3\}$:



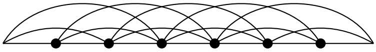
But they are all the same!

If G is a graph, the <u>power</u> $G^{(k)}$ has

- same vertex set $V(G)$
- an edge xy whenever $d_G(x, y) \leq k$.



P_∞



$P_\infty^{(3)}$

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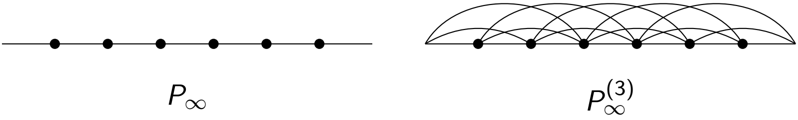

Lemma

For any finite generating sets S_1, S_2 of G, there exists $c \in \mathbb{N}$,

$$Cay(G, S_1) \subset \left(Cay(G, S_2)\right)^{(c)}.$$

Geometers say: all Cayley graphs are quasi-isometric.

Tree-width powers

For bounded degree graphs,

$$G \text{ has finite tree-width} \iff G^{(k)} \text{ has finite tree-width.}$$

So if any Cayley graph of Γ has finite tree-width,
then all of them have finite tree-width.

(For bounded degree graphs, finite tree-width is a quasi-isometric invariant.)

What now?

Tree-width of Cayley graphs is well understood.

Theorem (Kuske & Lohrey, '05)

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Which graph parameters can we look at next?
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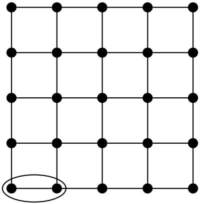
- Hadwiger's number (max size of K_t-minor)?
  Not stable under powers: if G is the infinite grid,
  G is planar, but $G^{(2)}$ contains K_∞ as minor!

Twin-width

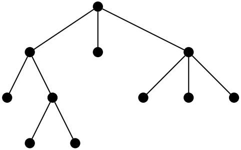
Twin-width:

- pick two vertices (can be non-adjacent) and contract them
- repeat until there is only one vertex left
- cost of the <u>contraction sequence</u> = max degree
- $\text{tww}(G)$ = min cost of a contraction sequence

(This definition only works for bounded degree)



$\text{tww} = 4$



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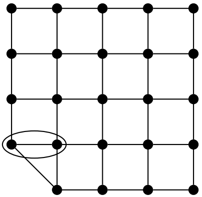

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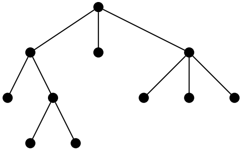

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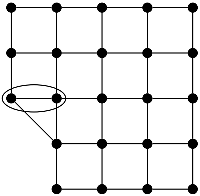

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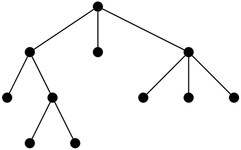

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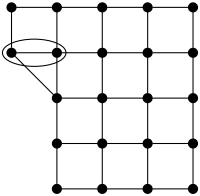

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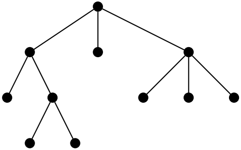

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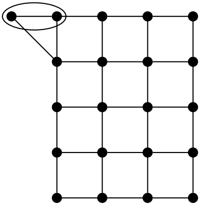

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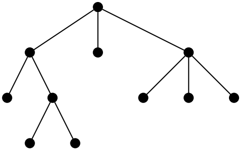

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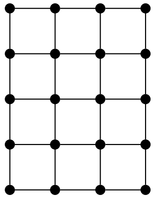

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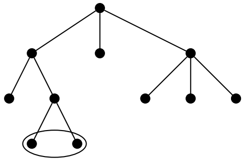

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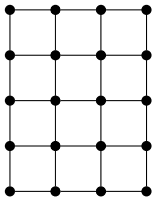

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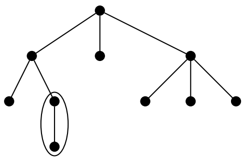

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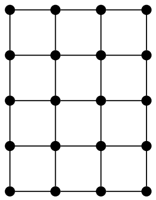

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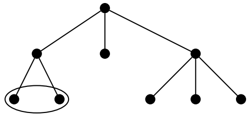

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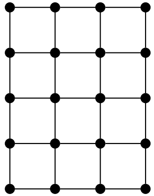

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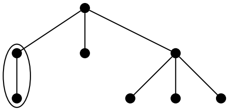

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Twin-width and quasi-isometries

Equivalent definition of twin-width:

- start with the partition into singletons $\mathcal{P} = \{\{x\} \ : \ x \in V(G)\}$
- merge two parts of $\mathcal{P}$
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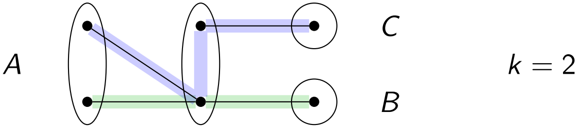
For $\mathcal{P}$ partition of $V(G)$, $G^{(k)}/\mathcal{P} \subseteq (G/\mathcal{P})^{(k)}$.



Thus $\text{tww}(G^{(k)}) \leq \text{tww}(G)^k$

Groups with finite twin-width

Summarizing:

Lemma

For any finite generating sets S_1, S_2 of Γ,

$$\text{tww}(Cay(\Gamma, S_1)) < \infty \iff \text{tww}(Cay(\Gamma, S_2)) < \infty.$$

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- all hyperbolic groups (contained in cartesian products of trees)

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bounded degree graphs with infinite twin-width exist,
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The number of 3-regular graphs on n vertices is $\sim (n/2)!$
Asymptotically, $(n/2)! \gg c^n$ for any constant c

Groups with infinite twin-width

Theorem (Osajda '20)

Let $(G_n)_{n \in N}$ *be a sequence of finite graphs with*

- *bounded degree,*
- *bounded* $\operatorname{diam}(G_n)/\operatorname{girth}(G_n)$ *ratio,*
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Then $(G_n)_{n \in N}$ *embeds isometrically into some Cayley graph.*

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- $\Rightarrow$ There is a sequence of graphs with unbounded twin-width satisfying the conditions

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- A non-neglectable proportion of bounded degree graphs satisfy the conditions of the theorem
- $\Rightarrow$ There is a sequence of graphs with unbounded twin-width satisfying the conditions
- $\Rightarrow$ The theorem gives a group with infinite twin-width

More groups with finite twin-width!

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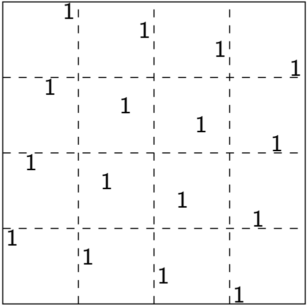
Cayley graphs / contraction sequences are not that convenient to prove finite twin-width. . .

Theorem (Bonnet, Eun-Jung Kim, Thomassé, Watrigant '20)

G a bounded degree graph. $\mathrm{tww}(G)$ *is bounded iff there is an ordering of $V(G)$ where the adjacency matrix has no k-grid as submatrix for some k.*



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Lemma

The following are equivalent:

- *$Cay(\Gamma, S)$ has finite twin-width*
- *$\forall s \in S, \exists k, M_s^<$ has no k-grid*
- *$\forall x \in \Gamma, \exists k, M_x^<$ has no k-grid*

Sketch: the matrix of $\mathrm{Cay}(\Gamma, S)$ is $\bigcup_{s \in S} M_s^<$.

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An ordering $<$ of Γ is <u>right invariant</u> if $\forall x, y, z \in \Gamma$

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Examples:

- natural order on $\mathbb{Z}$ is invariant
- lexicographic order on $\mathbb{Z}^n$ is invariant
- the free group $\mathbb{F}(a, b)$ has an invariant order

Non-examples:

- cyclic groups $\mathbb{Z}/n\mathbb{Z}$ is not orderable
- there exist non-orderable groups which do not contain any $\mathbb{Z}/n\mathbb{Z}$

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Lemma

Any orderable group has finite twin-width.

Take $<$ right-invariant ordering of Γ
For any $s \in \Gamma$, $x \mapsto xs$ is an increasing map, so $M_s^<$ has no 2-grid.

Uniform twin-width

Summary: Γ has finite twin-width $\iff$
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Recall the previous example: Take $<$ right-invariant ordering of Γ
For any $s \in \Gamma$, $x \mapsto xs$ is an increasing map, so $M_s^<$ has no 2-grid.
$\Rightarrow$ orderable groups have uniform twin-width 2

Uniform twin-width

Uniform twin-width is very nice to construct groups of finite twin-width! For example:

Lemma (group extension)

Let H be a (normal) subgroup of G. If H and G/H have uniform twin-width k, then so does G.

This gives a lot of finite twin-width groups:
for instance solvable groups (= constructed starting from commutative groups by extensions)

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Thank you!