Transductions and Variants of Twin-Width

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Twin-width

Twin-width is a 'graph complexity parameter':

$$\text{tww} : \mathrm{Graphs} \to \mathbb{N}.$$

Theorem (BKTW '20)

There is an algorithm which given

- *a graph G with n vertices,*
- *a witness that $\text{tww}(G) \leq d$, and*
- *a first-order formula ϕ*

tests if $G \models \phi$ in time $f(\phi, d) \cdot n$.

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Outline

Theorem (BKTW '20)

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This talk:
simple presentation of the proof, working for variants of twin-width.

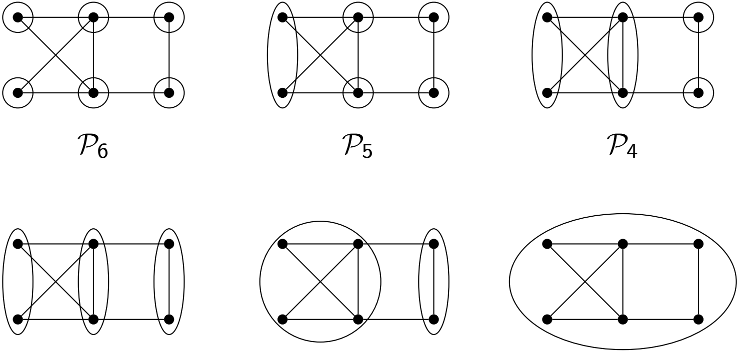
Ingredients:

- (standard) Gaifman's locality
- 'rank-twin-width'
- flips

Definition

Contraction sequence: $\mathcal{P}_n, \ldots, \mathcal{P}_1$ partitions of $V(G)$.

- $\mathcal{P}_n = \{\{v\} : v \in V(G)\}$ partition into singletons
- $\mathcal{P}_1 = \{V(G)\}$ trivial partition
- $\mathcal{P}_i$ is obtained by merging two parts of $\mathcal{P}_{i+1}$.



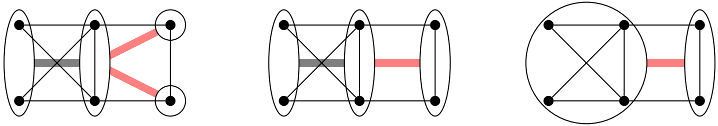
$\mathcal{P}_6$ $\mathcal{P}_5$ $\mathcal{P}_4$

$\mathcal{P}_3$ $\mathcal{P}_2$ $\mathcal{P}_1$

Definition

$$\mathcal{P} \text{ partition of } V(G) \quad \longrightarrow \quad \text{set } R_{\mathcal{P}} \subseteq \binom{\mathcal{P}}{2} \text{ of } \textit{red edges}$$

$XY \in R_{\mathcal{P}}$ when $X, Y \in \mathcal{P}$ are not *homogeneous*,
i.e. there is both an edge and a non-edge between them.



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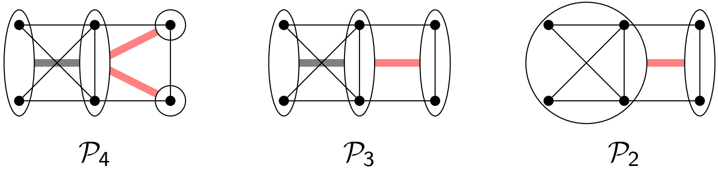

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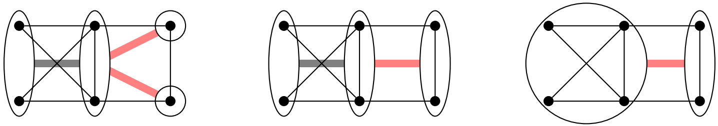

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'cost' of $\mathcal{P} = \Delta(R_{\mathcal{P}})$ (maximum degree in the 'red' graph $(\mathcal{P}, R_{\mathcal{P}})$)
'cost' of the contraction sequence $\mathcal{P}_n, \ldots, \mathcal{P}_1 = \max_i \Delta(R_{\mathcal{P}_i})$
$\mathrm{tww}(G) =$ minimum cost of a contraction sequence.

Strategy

Transduction: for $\phi(x, y)$ a first-order formula,

$$\phi(G) \coloneqq (V(G), \{xy : G \models \phi(x, y)\}).$$

Theorem

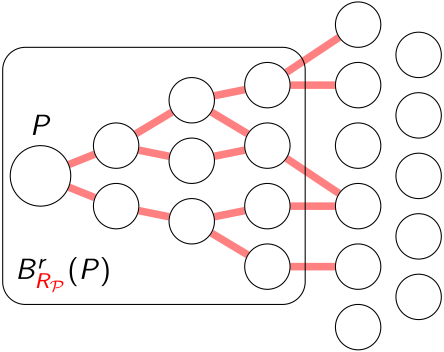
$$\mathrm{tww}(\phi(G)) \leq f(\mathrm{tww}(G)).$$

Strategy: given $\mathcal{P}_n, \ldots, \mathcal{P}_1$ contraction sequence, show

$$\text{cost of } \mathcal{P}_i \text{ in } \phi(G) \quad \leq \quad f(\text{cost of } \mathcal{P}_i \text{ in } G).$$

. . . almost!
We will need to adjust the definition of twin-width.

Locality



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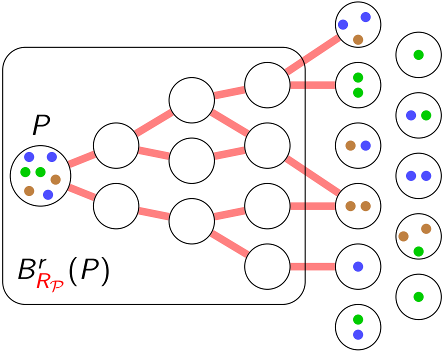
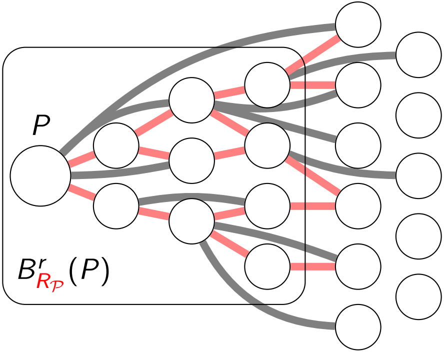


Theorem (Gaifman)

For any FO formula $\phi(x, y)$, there are $r, k \in \mathbb{N}$ such that any graph G has a colouring $\lambda : V(G) \to \{1, \ldots, k\}$ so that if $d_G(u, v) > r$, then $\phi(x, y)$ only depends on $\lambda(u), \lambda(v)$.

The radius r only depends on the quantifier rank of ϕ.

Locality



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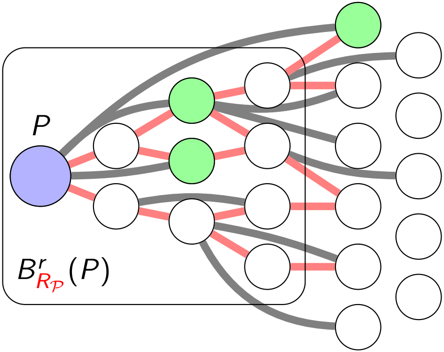

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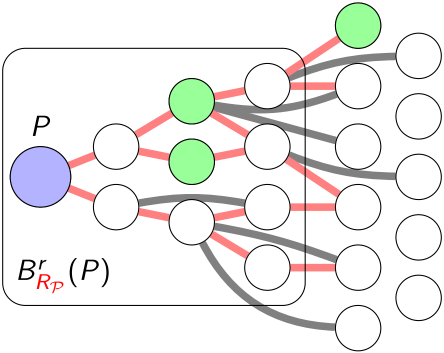
Locality

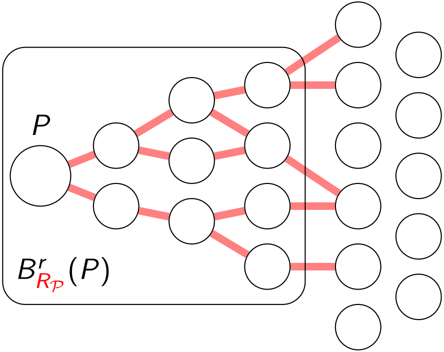


$A = P$ and $B = N_{\text{black}}(P)$ is a biclique

- Remove the biclique AB.
- Add unary predicates to G to mark A and B.
- In ϕ, replace $E(x, y)$ by $E(x, y) \vee (x \in A \wedge y \in B)$.

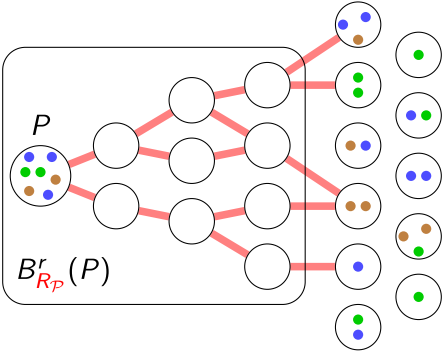
This increases the number k of colours given by Gaifman's theorem, but not the radius r!

Locality



Repeat for each part $Q \in B^r_{R_{\mathcal{P}}}(P)$ (bounded number)

Locality



By Gaifman's theorem, for $x \in P$, $y \notin B^r_{R_\mathcal{P}}(P)$, $\phi(x, y)$ only depends on the colours of x and y.

So we obtain:

Fact

In $\phi(G)$, the adjacency matrix of P versus $V \setminus B^r_{R_\mathcal{P}}(P)$ has bounded rank.

Summary:

Given $\mathcal{P}_n, \dots, \mathcal{P}_1$ contraction sequence for G
and $R_n, \dots, R_1$ the corresponding red edges

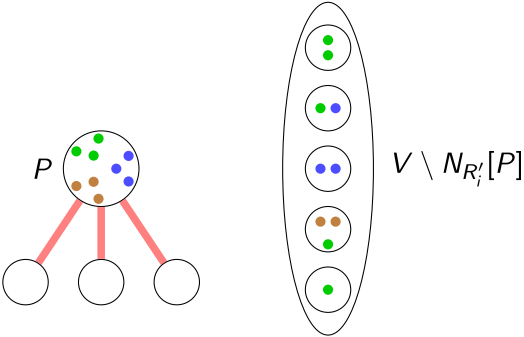
We define (with $r =$ Gaifman radius of ϕ)

$$R_i' = \{PQ : d_{R_i}(P, Q) \leq r\}.$$

- Degree is still bounded: $\Delta(R_i') \leq (\Delta(R_i))^r$
- In $\phi(G)$ we proved that for all $P \in \mathcal{P}_i$, the rank of P versus $V \setminus N_{R_i'}[P]$ in $\phi(G)$ is bounded

Going back to twin-width

The rank of P versus $V \setminus N_{R_i'}[P]$ is at most k



Going back to twin-width

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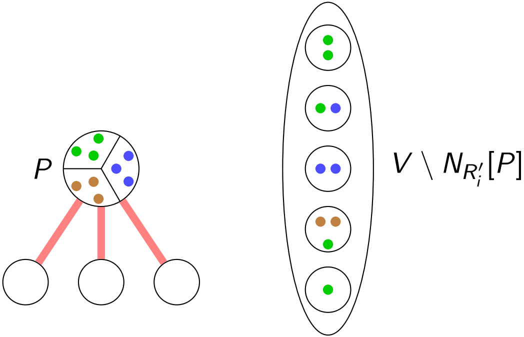

Ignoring $N_{R'_i}(P)$, split P into neighbourhood types $P_1, \dots, P_{2^k}$

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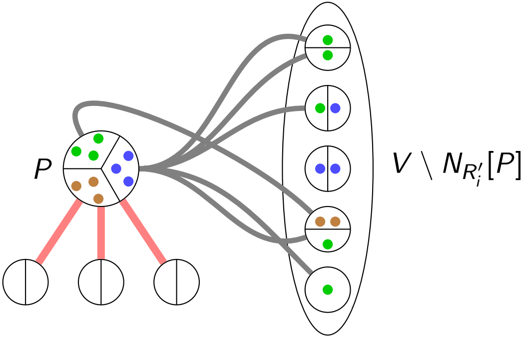

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Repeat for all parts of $\mathcal{P}_i$:
parts non-adjacent in R'_i are now fully adjacent or non-adjacent

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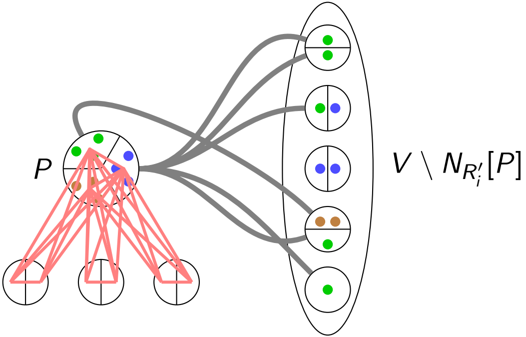

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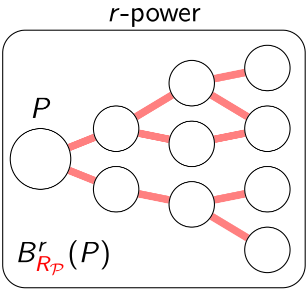
Transformations of red graphs



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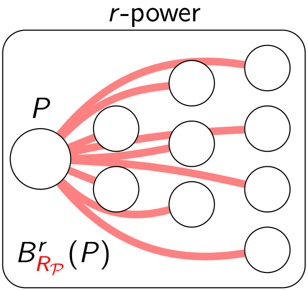
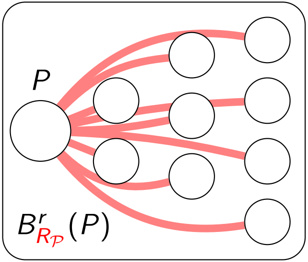
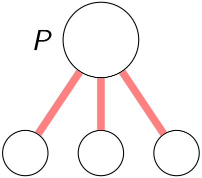


Transformations of red graphs

r-power



k-blowup



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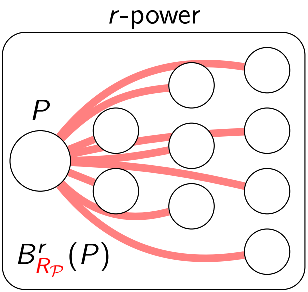

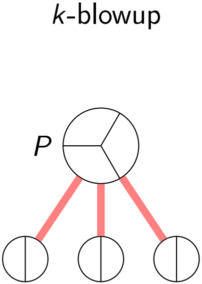

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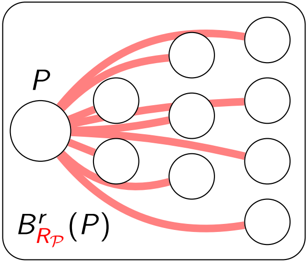

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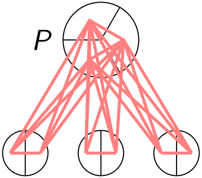

Variants of twin-width

Consider any graph parameter $\mathfrak{p} : \mathrm{Graphs} \to \mathbb{N}$.
'reduced-$\mathfrak{p}$' is defined like twin-width, replacing

the red graphs R_i have bounded degree

by

the red graphs R_i have bounded degree *and bounded* $\mathfrak{p}$

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Theorem

If having bounded $\mathfrak{p}$ is preserved by taking subgraphs, bounded powers, and bounded blowups, then having bounded reduced-$\mathfrak{p}$ is preserved by FO transductions.

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Theorem

If having bounded $\mathfrak{p}$ is preserved by FO transductions between bounded degree graphs, then having bounded reduced-$\mathfrak{p}$ is preserved by FO transductions.

This works for $\mathfrak{p}$ being tree-width, path-width, band-width, ...