Twin-Width of Groups and Graphs of Bounded Degree

Édouard Bonnet Colin Geniet
Romain Tessera Stéphan Thomassé

18 Mars 2022

Twin-Width

Contractions:

- Any pair of vertices can be contracted (not just edges)
- Loops and double edges are removed

Contraction sequence: $G_n, \ldots, G_1$, where

- G_i is result of a contraction in G_{i+1}
- G_1 has just one vertex

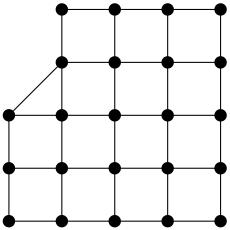
Twin-width:

$$\mathrm{tww}(G) = \min_{\substack{G=G_n,\ldots,G_1 \\ \text{contr. seq.}}} \max_{\substack{i\in[n] \\ v\in V(G_i)}} d_{\text{red}}(v)$$

Simplified definition for graphs of bounded degree.

Examples

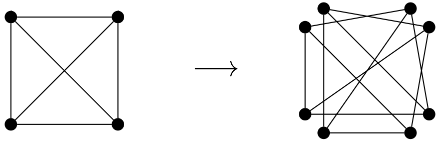
- Paths, cycles have $\mathrm{tww} = 2$
- Trees have $\mathrm{tww} = \Delta$
- Grids have $\mathrm{tww} = 4$
- d-dimensional grids have $\mathrm{tww} = O(d)$



Example: Bilu–Linial Expanders

2-lift of G:

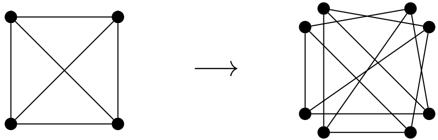
- Duplicate each vertex $v \in V(G)$ into v_0, v_1.
- For $uv \in EG$ add either
  - the edges $u_0 v_0$ and $u_1 v_1$ (straight),
  - or the edges $u_0 v_1$ and $u_1 v_0$ (crossing).



Example: Bilu–Linial Expanders

2-lift of G:

- Duplicate each vertex $v \in V(G)$ into v_0, v_1.
- For $uv \in EG$ add either
  - the edges $u_0 v_0$ and $u_1 v_1$ (straight),
  - or the edges $u_0 v_1$ and $u_1 v_0$ (crossing).



Theorem (Bilu and Linial, '06)

Iterated 2-lifts starting from K_4, with random choices of straight/crossing, yield cubic expanders almost surely.

All iterated 2-lifts of K_4 have $\mathrm{tww} \leq 6$: reverse the lift sequence.

Why Twin-Width

For classes of graphs with bounded twin-width:

- FPT first-order model checking (given a contraction sequence) [É.B., E.J. Kim, S.T., R.Watrigant].
- Quasi-polynomially χ-bounded [Mi.Piliczuk,M.Sokołowski]
- Some FPT and approximation algorithms for independent set, dominating set [É.B., C.G., E.J. Kim, S.T., R.Watrigant].

Small Classes

When counting graphs on n vertices in a class $\mathcal{C}$, we count graphs in $\mathcal{C}$ with vertices labeled from 1 to n.

A class is *small* if the number of graphs on n vertices is

$$O(n! \cdot c^n) = 2^{n \log n + O(n)}$$

Examples:

- Trees
- Proper minor-closed classes [Norine, Seymour, Thomas, Wollan]

Theorem (É.B., C.G., E.J. Kim, S.T., R.Watrigant)

Any class with bounded twin-width is small.

Not Small Classes

Number of cubic graphs on n vertices:

$$2^{3/2 \cdot n \log n + \Omega(n)}$$

Number of graphs of twin-width k on n vertices:

$$2^{n \log n + O_k(n)}$$

Corollary

Expected twin-width of random cubic graphs is unbounded.

Questions

1. Can we find explicit constructions of graphs with bounded degree and unbounded twin-width?
2. Do all small (hereditary) classes have bounded twin-width?

Power of Graphs

The kth power of G is the graph $G^{(k)}$ with

- vertices $V(G)$
- an edge xy whenever $d_G(x, y) \leq k$

Lemma

$$\mathrm{tww}\left(G^{(k)}\right) \leq \mathrm{tww}(G)^k$$

Generalisation (for the general definition of twin-width):

Theorem

For any first-order transduction Φ and graph G,

$$\mathrm{tww}(\Phi(G)) \leq f(\mathrm{tww}(G), \Phi)$$

Power of Graphs (Proof)

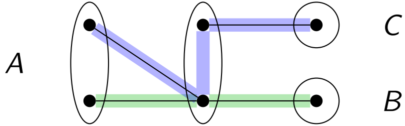
Contraction sequence of width t:

$$G = G_n, \ldots, G_1 = K_1$$

same sequence on $G^{(k)}$:

$$G^{(k)} = G'_n, \ldots, G'_1 = K_1$$

G'_i is a subgraph of $G_i^{(k)}$:



$k = 2$

$$\Delta(G'_i) \leq \Delta\left(G_i^{(k)}\right) \leq \Delta(G_i)^k \leq t^k$$

Coarse Geometry

$f : X \to Y$ is a λ-quasi-isometric embedding if

$$\lambda^{-1} d_X(x, y) - \lambda \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \lambda$$

Coarse Geometry

$f : X \to Y$ is a λ-quasi-isometric embedding if

$$\lambda^{-1} d_X(x, y) - \lambda \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \lambda$$

Lemma

If $f : H \to G$ is a λ-quasi-isometric embedding of graphs of bounded degree,

$$\mathrm{tww}(H) \leq f(\lambda, \mathrm{tww}(G))$$

Coarse Geometry

$f : X \to Y$ is a λ-quasi-isometric embedding if

$$\lambda^{-1} d_X(x, y) - \lambda \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \lambda$$

Lemma

If $f : H \to G$ is a λ-quasi-isometric embedding of graphs of bounded degree,

$$\mathrm{tww}(H) \leq f(\lambda, \mathrm{tww}(G))$$

For G infinite, define

$$\mathrm{tww}(G) = \sup_{H \subset_{\mathsf{fin}} G} \mathrm{tww}(H)$$

For infinite graphs with bounded degree, finite twin-width is preserved by quasi-isometries.

Cayley Graphs

Let Γ group generated by S finite.
The Cayley graph $\mathrm{Cay}(\Gamma, S)$ has

- vertices Γ
- an edge from x to xs for every $x \in \Gamma$, $s \in S$.

Cayley Graphs

Let Γ group generated by S finite.
The Cayley graph $\mathrm{Cay}(\Gamma, S)$ has

- vertices Γ
- an edge from x to xs for every $x \in \Gamma$, $s \in S$.

Examples:

- $\mathrm{Cay}(\mathbb{Z}, \{1\})$ is the infinite path
- $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{1\}) = C_n$
- $\mathrm{Cay}(\mathbb{Z}^2, \{(0,1),(1,0)\})$ is the infinite grid (d-dimentional grid for $\mathbb{Z}^d$)
- If $\mathbb{F}(S)$ is the group freely generated by S, $\mathrm{Cay}(\mathbb{F}(S), S)$ is the $2|S|$-regular tree.

Twin-Width of Groups

Lemma

All Cayley graphs of Γ are quasi-isometric.

Finite twin-width is well-defined on groups.

Examples:

- $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$
- Free groups
- Products of groups with finite twin-width
- (Finitely generated) commutative groups

Cayley Graphs

Let Γ group generated by S.
Let $\mathcal{C}$ be the class of finite induced subgraphs of $\mathrm{Cay}(\Gamma, S)$.

Lemma

$\mathcal{C}$ is small.

Proof.

Any $G \in \mathcal{C}$ is characterized by a directed spanning tree, with edges labelled with $S \cup S^{-1}$. □

Cayley Graphs

Let Γ group generated by S.
Let $\mathcal{C}$ be the class of finite induced subgraphs of $\mathrm{Cay}(\Gamma, S)$.

Lemma

$\mathcal{C}$ is small.

Proof.

Any $G \in \mathcal{C}$ is characterized by a directed spanning tree, with edges labelled with $S \cup S^{-1}$. $\square$

Suppose Γ has infinite twin-width.

- $\mathcal{C}$ is class of graphs with bounded degree and unbounded twin-width
- $\mathcal{C}$ is a small class of graphs with unbounded twin-width

Group with Infinite Twin-Width

Theorem (Osajda, 2020)

Let $(G_n)_{n \in N}$ *be a sequence of graphs with*

- $\Delta(G_n) \leq D$
- $diam(G_n)/girth(G_n) \leq A$
- $girth(G_{n+1}) \geq girth(G_n) + 6$

There exists a group Γ *finitely generated by* S *such that* $Cay(\Gamma, S)$ *contains all* G_n *as isometric subgraphs.*

Group with Infinite Twin-Width

Theorem (Osajda, 2020)

Let $(G_n)_{n \in N}$ be a sequence of graphs with

- $\Delta(G_n) \leq D$
- $diam(G_n)/girth(G_n) \leq A$
- $girth(G_{n+1}) \geq girth(G_n) + 6$

There exists a group Γ finitely generated by S such that $Cay(\Gamma, S)$ contains all G_n as isometric subgraphs.

Lemma

There exists graphs G with arbitrarily large twin-width, and

- $\Delta(G) \leq 6$
- $diam(G) \leq 3 \log n$
- $girth(G) \geq \frac{\log n}{K}$

Lemma

There exists graphs G with arbitrarily large twin-width, and

- $\Delta(G) \leq 6$
- $diam(G) \leq 3 \log n$
- $girth(G) \geq \frac{\log n}{K}$

Lemma

There exists graphs G with arbitrarily large twin-width, and

- $\Delta(G) \leq 6$
- $diam(G) \leq 3 \log n$
- $girth(G) \geq \frac{\log n}{K}$

Sketch:

- Start from a random cubic graph.

Lemma

There exists graphs G with arbitrarily large twin-width, and

- $\Delta(G) \leq 6$
- $diam(G) \leq 3 \log n$
- $girth(G) \geq \frac{\log n}{K}$

Sketch:

- Start from a random cubic graph.
- With probability $> \frac{1}{2}$, there are not too many short ($< \frac{\log n}{K}$) cycles. Cut these short cycles (remove an edge in each).

Lemma

There exists graphs G with arbitrarily large twin-width, and

- $\Delta(G) \leq 6$
- $diam(G) \leq 3 \log n$
- $girth(G) \geq \frac{\log n}{K}$

Sketch:

- Start from a random cubic graph.
- With probability $> \frac{1}{2}$, there are not too many short $(< \frac{\log n}{K})$ cycles. Cut these short cycles (remove an edge in each).
- Choose a maximum packing X of vertices at distance pairwise $> \log n$, and join them with a balanced cubic tree.

Lemma

There exists graphs G with arbitrarily large twin-width, and

- $\Delta(G) \leq 6$
- $diam(G) \leq 3 \log n$
- $girth(G) \geq \frac{\log n}{K}$

Sketch:

- Start from a random cubic graph.
- With probability $> \frac{1}{2}$, there are not too many short ($< \frac{\log n}{K}$) cycles. Cut these short cycles (remove an edge in each).
- Choose a maximum packing X of vertices at distance pairwise $> \log n$, and join them with a balanced cubic tree.
- The graph obtained satisfies the first 3 conditions.

Lemma

There exists graphs G with arbitrarily large twin-width, and

- $\Delta(G) \leq 6$
- $diam(G) \leq 3 \log n$
- $girth(G) \geq \frac{\log n}{K}$

Sketch:

- Start from a random cubic graph.
- With probability $> \frac{1}{2}$, there are not too many short ($< \frac{\log n}{K}$) cycles. Cut these short cycles (remove an edge in each).
- Choose a maximum packing X of vertices at distance pairwise $> \log n$, and join them with a balanced cubic tree.
- The graph obtained satisfies the first 3 conditions.
- The above requires only $n^{1-\epsilon}$ edge editions. This implies that the class of graphs satisfying the first 3 conditions is not small.

End Result

- There is a group with infinite twin-width.
- We have no idea what it looks like.
- It doesn't help with constructing graphs of bounded degree and unbounded twin-width.
- There is a small class of graphs with unbounded twin-width.

Grid Characterisation

A k-grid is a $k \times k$-division in which every zone has a '1'.

0	0	0	1	0
0	1	0	0	1
0	0	0	1	1
1	0	1	0	0
0	1	1	0	1

Grid number = maximum size of a grid.

Theorem

- *A matrix M has bounded twin-width if and only if it has bounded grid number.*
- *A graph G has bounded twin-width if and only if there is an order $<$ on $V(G)$ such that the adjacency matrix of G has bounded grid number.*

Grid Characterisation for Groups

For $x \in \Gamma$, $<$ order on Γ, $M_x^<$ is the permutation matrix of

$$(y \in \Gamma) \mapsto y \cdot x$$

Claim

The adjacency matrix of $Cay(\Gamma, S)$ *with order* $<$ *is*

$$\bigvee_{s \in S \cup S^{-1}} M_s^<$$

Lemma

Γ *has finite twin-width if and only if there is an order* $<$ *on* Γ *such that for every* $x \in \Gamma$, $M_x^<$ *has finite grid number.*

Matrix Definition

Definition

Γ has finite twin-width if there is an order $<$ on Γ such that for every $x \in \Gamma$, $M_x^<$ has finite twin-width.

This definition works for non finitely generated groups.

Matrix Definition

Definition

Γ has finite twin-width if there is an order $<$ on Γ such that for every $x \in \Gamma$, $M_x^<$ has finite twin-width.

This definition works for non finitely generated groups.

Definition

Uniform twin-width is

$$\mathrm{utww}(\Gamma) = \inf_{<\text{ order on }\Gamma} \sup_{x \in \Gamma} \mathrm{tww}(M_x^<)$$

Uniform Twin-Width

Lemma

If G is a group, $H \leq G$ a subgroup

$$\mathrm{utww}(G) \leq \max(\mathrm{utww}(H), \mathrm{utww}(G/H))$$

Groups with finite uniform twin-width:

- Ordered groups
- Finitely generated abelian groups.
- Polycyclic groups
- Polynomial growth

Open Questions

- Explicit construction for groups (or graphs of bounded degree) with infinite twin-width?
- Separating twin-width and uniform twin-width for groups?
- Is there a universal bound on uniform twin-width of finite groups?

Open Questions

- Explicit construction for groups (or graphs of bounded degree) with infinite twin-width?
- Separating twin-width and uniform twin-width for groups?
- Is there a universal bound on uniform twin-width of finite groups?

3-dim. grid with diagonals has infinite stack number [Eppstein et. al.]
Stack number is not a group invariant, but queue number is!

- Matrix characterisation, uniform variants adapt to queue number.
- Separating queue number and twin-width?