Tournaments, First-Order Logic, and Twin-Width

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Simple classes for FO

Setting: $\mathcal{C}$ class of graphs (or relational structures), hereditary ($=$ closed under induced subgraphs).

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$\mathcal{C}$ has *fixed-parameter tractable (FPT) FO model checking* if there is an algorithm to test $G \models \phi$ for $G \in \mathcal{C}$, in time

$$f(\phi) \cdot poly(|G|)$$

NIP and FO model checking

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Theorem (Grohe, Kreutzer, Siebertz & Adler, Adler)

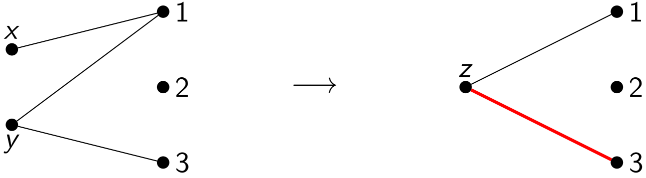
Let $\mathcal{C}$ class of graphs closed under subgraphs. TFAE:

- *$\mathcal{C}$ is dependent,*
- *$\mathcal{C}$ has FPT model checking,*
- *$\mathcal{C}$ is nowhere dense.*

Twin-width

Contracting vertices x, y into z:

- if both xu, yu are edges, then zu is an edge
- if neither xu, yu are edges, then zu is not an edge
- if exactly one of xu, yu are edges, then zu is an error edge



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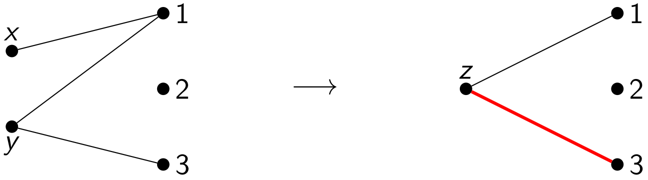

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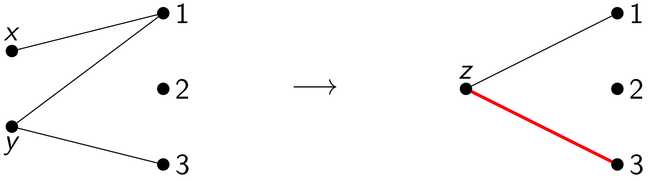
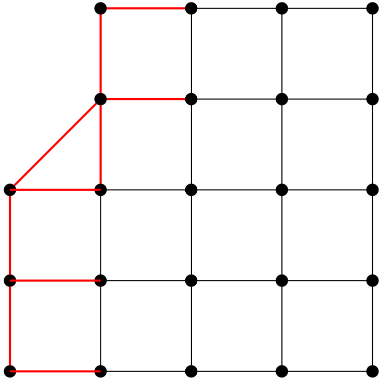


Contraction sequence: start from G, contract until only one vertex is left.

width of the sequence = maximum red degree

twin-width $\text{tww}(G)$ = minimum width of a sequence for G

Example: grids



FO and twin-width

Theorem (Bonnet, Kim, Thomassé, Watrigant, '20)

For any FO transduction Φ, there is a function f such that

$$\mathrm{tww}(\Phi(G)) \leq f(\mathrm{tww}(G))$$

Corollary

If $\mathcal{C}$ has bounded twin-width, it is NIP.

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Theorem (Bonnet, Kim, Thomassé, Watrigant, '20)

Given a graph G, a FO formula ϕ, and a contraction sequence of width t, one can test $G \models \phi$ in time $f(\phi, t) \cdot n$.

effectively bounded twin-width $\Rightarrow$ FPT model checking

Converse results?

Cubic graphs:

- are NIP,
- have FPT model checking,
- but do *not* have bounded twin-width (counting argument).

Twin-width and ordered graphs

Ordered graph $(G, <)$: graph G with an order $<$ on the vertices.

- FO logic can use the order:

$$\forall x, y, z, x < y < z \wedge E(x, z) \Rightarrow E(x, y)$$

- Out of order contractions create errors for twin-width.

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Theorem (BGOSTT, '21)

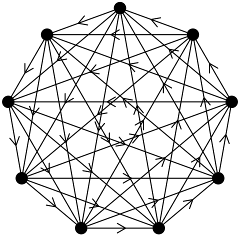
Twin-width of ordered graphs can be approximated up to some function, and witnesses of twin-width can be computed.

Furthermore, for $\mathcal{C}$ a hereditary class of ordered graphs, TFAE:

- *$\mathcal{C}$ is NIP,*
- *$\mathcal{C}$ has FPT FO model checking,*
- *$\mathcal{C}$ has bounded twin-width.*

Tournaments

Tournament: clique with a choice of orientation of each edge.



Twin-width for tournaments: edges in opposite directions cause errors.

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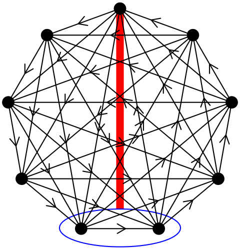

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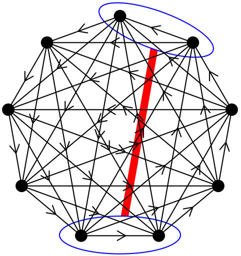

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Twin-width and tournaments

Theorem (G., T.)

There is a function f and an FPT algorithm which given a tournament T and $k \in \mathbb{N}$ answers

$$\operatorname{tww}(T) \geq k \qquad \text{or} \qquad \operatorname{tww}(T) \leq f(k)$$

Theorem (G., T.)

Let $\mathcal{C}$ be a hereditary class of tournaments. TFAE:

- *$\mathcal{C}$ is NIP,*
- *$\mathcal{C}$ has FPT FO model checking,*
- *$\mathcal{C}$ has bounded twin-width.*

Transducing a total order?

Is there a FO transduction which gives a total order on any tournament?

- If yes, tournaments are FO-equivalent to ordered graphs.
- NIP, FPT model checking, and bounded twin-width go through transductions.
- So the result on tournaments reduces to the result on ordered graphs.

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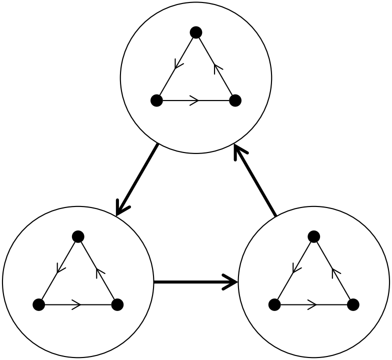
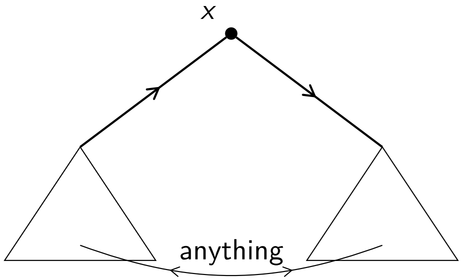
Counter-example:



Binary search trees in tournaments

If y is a $\begin{cases} \text{left} \\ \text{right} \end{cases}$ descendant of x, then $\begin{cases} y \to x \\ x \to y \end{cases}$



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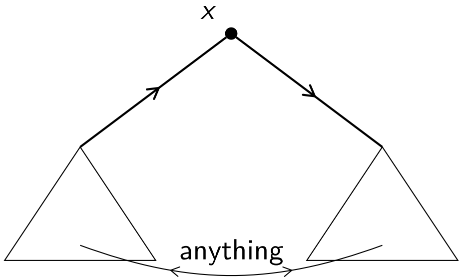

BST order: left-to-right order on a BST.

BST orders are good for twin-width

Lemma (G., T.)

There is a function f such that if T is a tournament, $<$ a BST order, then

$$\mathrm{tww}(T, <) \leq f(\mathrm{tww}(T))$$

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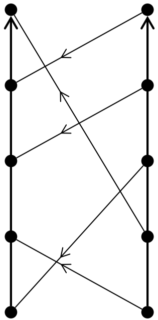
We will use:

Theorem (Bonnet et al., '21)

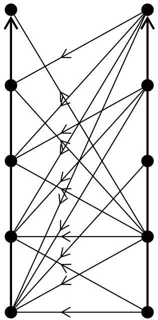
For an ordered graph $(G,<)$, TFAE:

- *$(G,<)$ has large twin-width*
- *The matrix of G in the order $<$ has a large rank minor*

Obstruction to twin-width



$\mathcal{F}_{=}(\sigma)$



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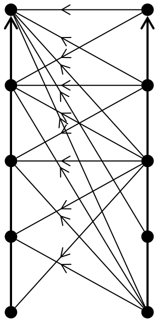

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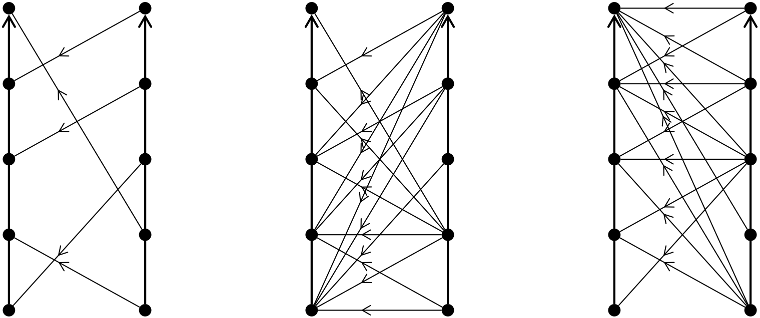

$\mathcal{F}_{=}(\sigma)$ $\mathcal{F}_{\leq}(\sigma)$ $\mathcal{F}_{\geq}(\sigma)$

Theorem (G., T.)

$\mathcal{C}$ hereditary class of tournaments has unbounded twin-width if and only if $\mathcal{C}$ contains one of $\mathcal{F}_{=}(\sigma), \mathcal{F}_{\leq}(\sigma), \mathcal{F}_{\geq}(\sigma)$ for any permutation σ.

Characterisation of NIP and FPT model checking

Lemma

The obstructions $\mathcal{F}_{=}(\sigma), \mathcal{F}_{\leq}(\sigma), \mathcal{F}_{\geq}(\sigma)$ encode arbitrary graphs in first-order logic.

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Lemma

The obstructions $\mathcal{F}_=(\sigma), \mathcal{F}_\leq(\sigma), \mathcal{F}_\geq(\sigma)$ encode arbitrary graphs in first-order logic.

Corollary

If $\mathcal{C}$ is a (hereditary) class of tournaments with unbounded twin-width, then

- *$\mathcal{C}$ is independent, and*
- *FO model checking in $\mathcal{C}$ is $\mathrm{AW}[*]$-complete.*

Generalisations

The results still hold

- for arbitrary binary relational structures, where one of the relations is a tournament, and
- when replacing tournaments with oriented graphs with bounded independence number.

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- when replacing tournaments with oriented graphs with bounded independence number.

We also obtain an enumerative characterisation:

Theorem

If $\mathcal{C}$ is a hereditary class of tournaments, TFAE:

- *$\mathcal{C}$ has bounded twin-width,*
- *$\mathcal{C}$ has growth at most c^n*
- *$\mathcal{C}$ has growth less than $(n/2-2)!$*

Conclusion

For tournaments, bounded twin-width, NIP, and FPT FO model checking are equivalent + algorithm and characterisation by forbidden structures.

Based on similar results for ordered graphs [BGOSTT '21]
Main tool: BST order

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For tournaments, bounded twin-width, NIP, and FPT FO model checking are equivalent + algorithm and characterisation by forbidden structures.

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Questions:

- NIP $\iff$ FPT model checking in general?
- Approximating twin-width in general?
- The equivalence 'NIP $\iff$ bounded twin-width' is called *delineation* [BCKKLT '22].
  - Interval graphs are delineated.
  - Conjectured to be delineated: segment graphs, some visibility graphs.