Maximum Independent Set when excluding induced minors: $K_1 + tK_2$ and $tC_3 \uplus C_4$

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Given a graph G, find a largest set S of pairwise non-adjacent vertices.

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Theorem

MIS is polynomial on H-minor free graphs iff H is planar.

- If H is planar, H-minor free graphs have bounded tree-width (grid minor theorem). MIS is polynomial by dynamic programming.
- If H is *not* planar, all planar graphs are H-minor free.

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Conjecture (Dallard, Milanič, Štorgel)

MIS is polynomial on induced H-minor free graphs iff H is planar.

Some known results

Known algorithms for MIS for some excluded induced minors: (not exhaustive!)

poly	quasi poly
P_6 (GKPP '22)	P_k (GL '20)
C_5 (ACPRS '20)	C_k (GLPPR '21)
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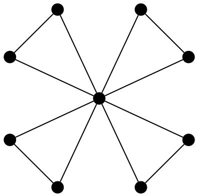
4-windmill:

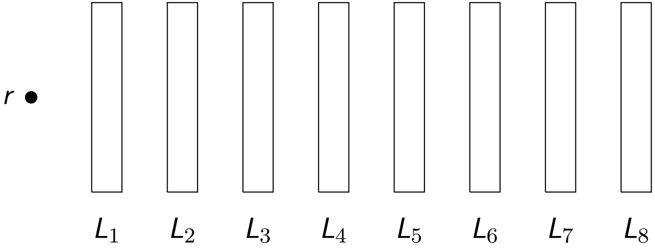


Theorem

MIS is polynomial on t-windmill induced-minor free graphs.

No t-windmill induced minor — bounded diameter case

BFS:



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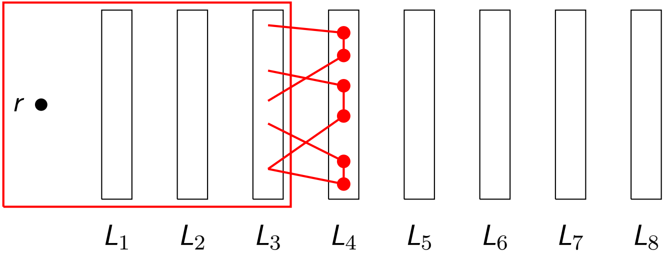

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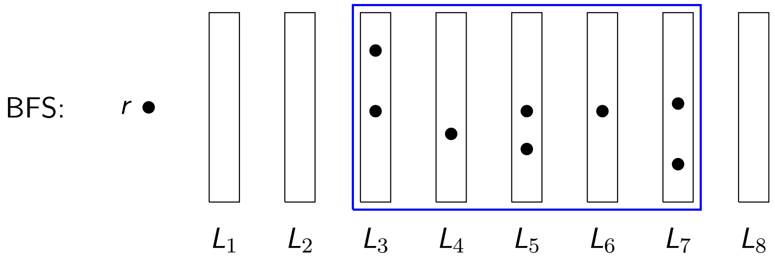

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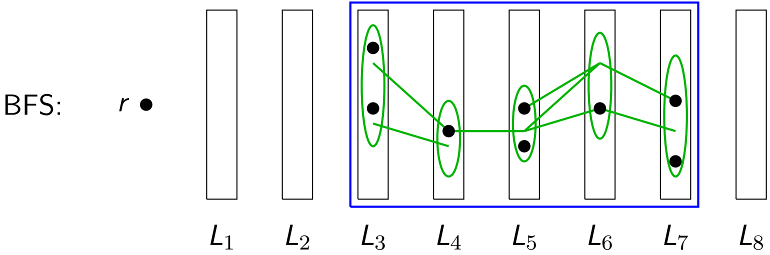

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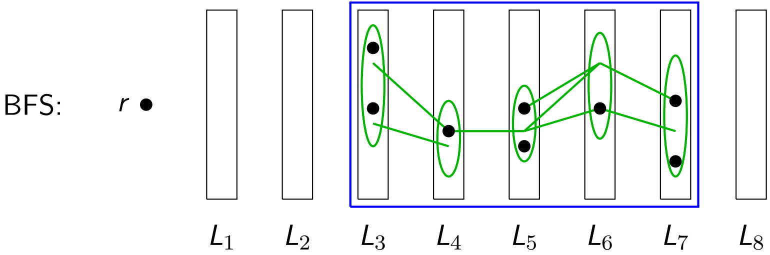

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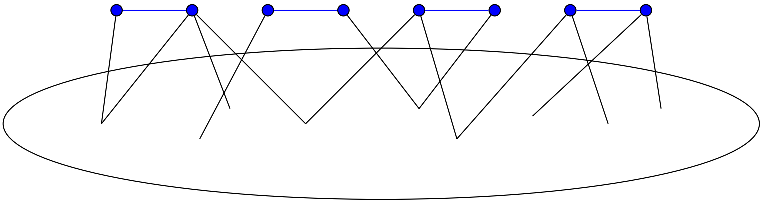
Guess a maximal IS $S_i \subseteq L_i$ such that $S^* \cap L_i \subseteq S_i$.
$\bigcup_i S_i$ is bipartite and contains S^*

No t-windmill induced minor

Consider a t-matching M (if there are none, Alekseev's result applies).



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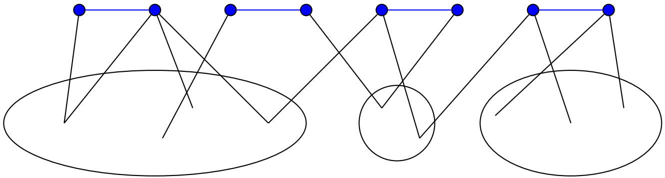

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Choose M to maximize the largest component of $G - M$.

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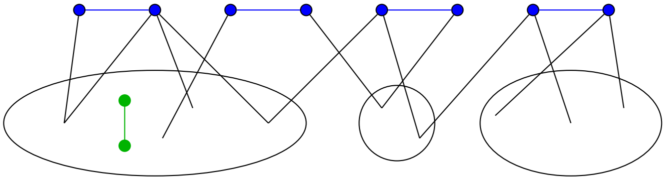
Let C be a component of $G - M$, $e \in C \setminus N[M]$.



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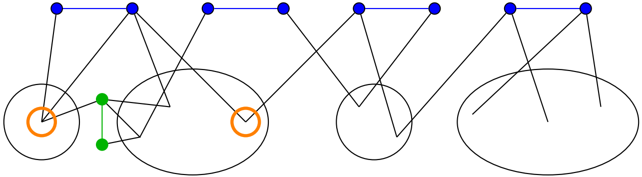
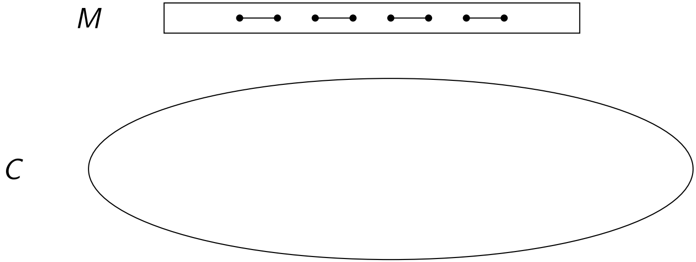


M is the good t-matching. C a component of $G - M$.



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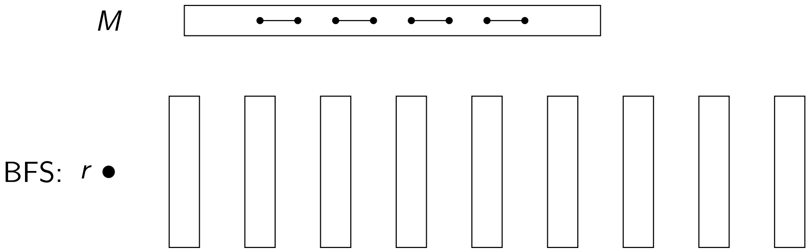

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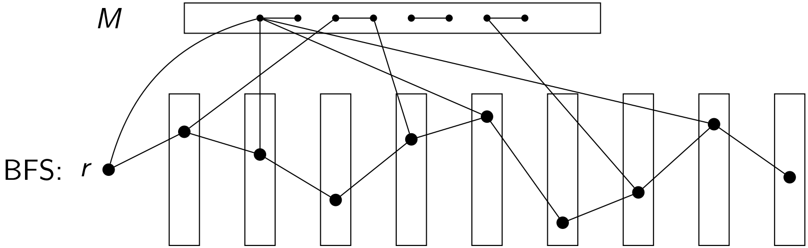

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An induced path intersects $N[M]$ at most $O(t^2)$ times.

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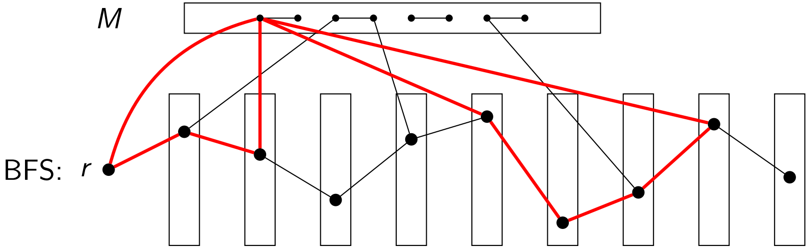

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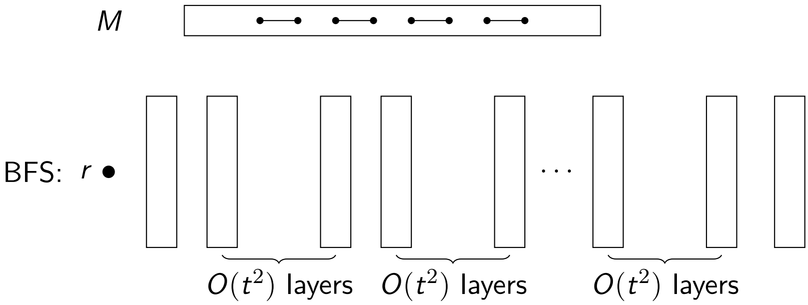

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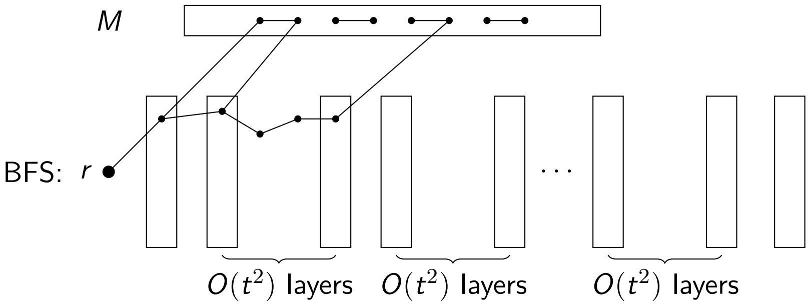

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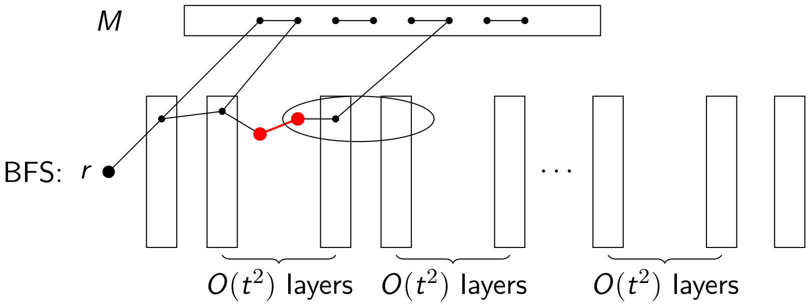

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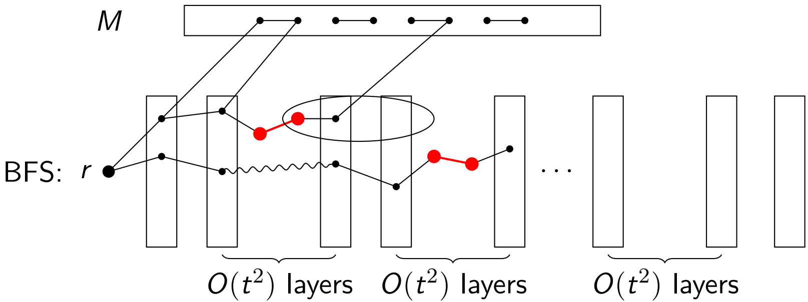

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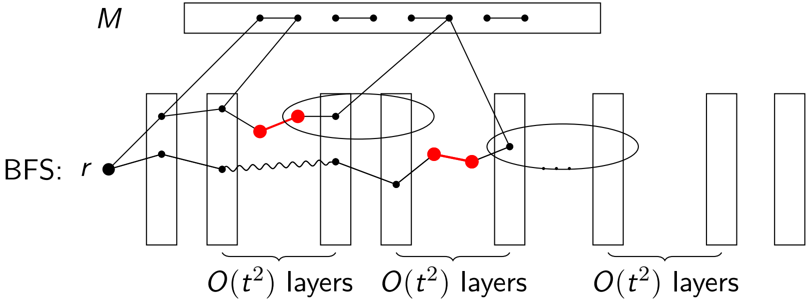

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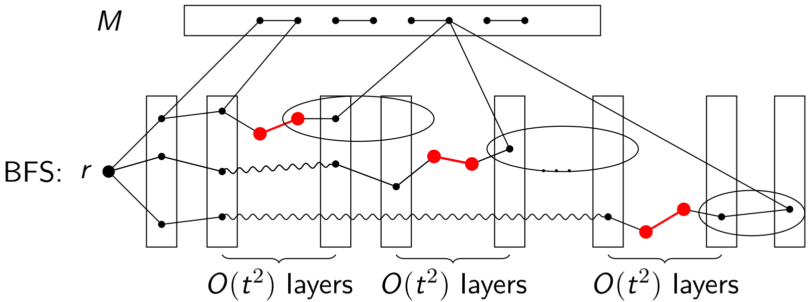

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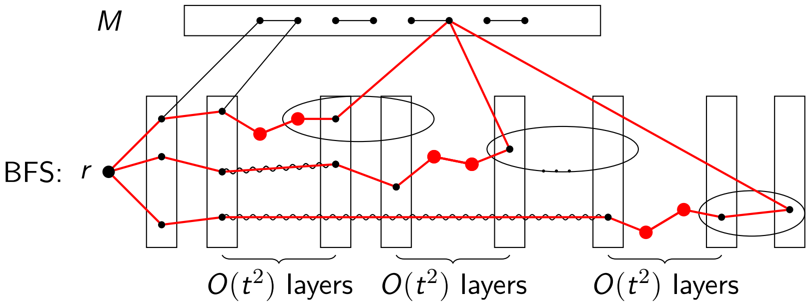

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$\Rightarrow$ path decomposition such that:

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Recall

Lemma

For t, l fixed, MIS is polynomial inside l consecutive layers.

$\Rightarrow$ we can solve MIS inside a bag of the path decomposition

Dynamic programming...

Open questions

- MIS with forbidden induced minors: P_7, C_6, ...

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