Factorising Pattern-Free Permutations

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Teaser for LoGAlg

Theorem (Folklore)

A class $\mathcal{C}$ has bounded cliquewidth iff $\mathcal{C}$ is CMSO transduction of trees.

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Theorem (Folklore)

A class $\mathcal{C}$ has bounded cliquewidth iff $\mathcal{C}$ is CMSO transduction of graphs of cliquewidth 1.

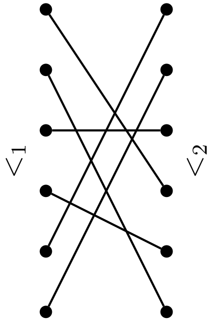
We prove:

Theorem

A class $\mathcal{C}$ has bounded twin-width iff $\mathcal{C}$ is FO transduction of graphs of twin-width c, for some universal constant c.

And Now for Something Completely Different

Permutation = two linear orders on the same set: $(X, <_1, <_2)$



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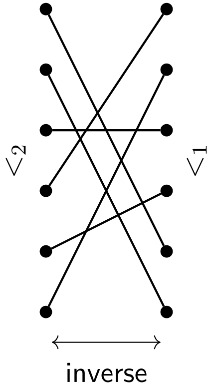

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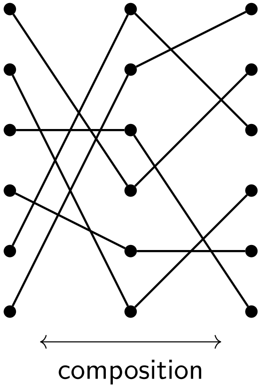

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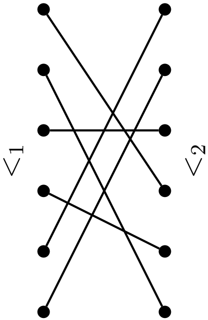
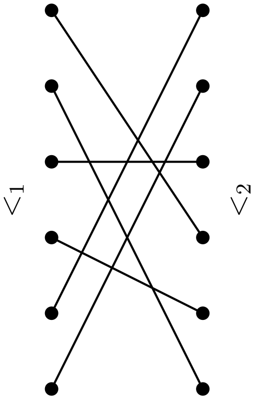


Theorem (BNdMST '21)

For any class $\mathcal{C}$ of bounded twin-width, there exists $\mathcal{C}'$ class of permutations of bounded twin-width such that $\mathcal{C}'$ transduces $\mathcal{C}$.

Patterns in permutations



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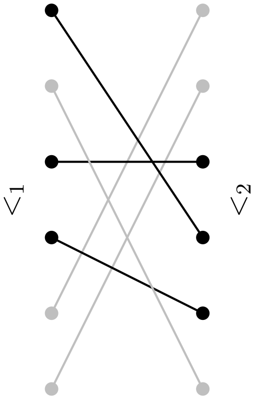

Permutation $(X, <_1, <_2)$

Pattern $=$ induced substructure

$$(Y, <_1, <_2), \quad Y \subset X$$

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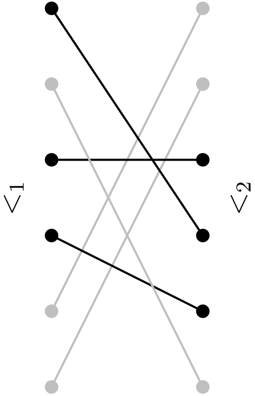

Permutation $(X, <_1, <_2)$

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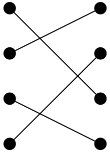
Pattern-free permutation class:

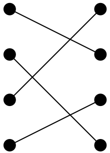
$$\mathcal{F}(\tau) = \{\sigma \ : \ \tau \not\subseteq \sigma\}$$

Separable permutations

Separable permutations $= \mathcal{F}(3142, 2413)$

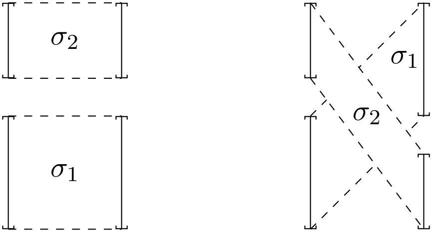
Forbidden patterns:





Separable permutations

Separable permutations $= \mathcal{F}(3142, 2413)$
$=$ permutations created by direct/skew sum



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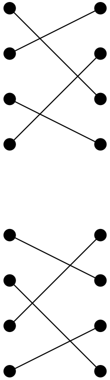

Pattern-free classes are nice

Theorem (Marcus–Tardos '04)

For any τ, there is a constant c such that $\mathcal{F}(\tau)$ has $\leq c^n$ permutations of size n.

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Recognition algorithm:

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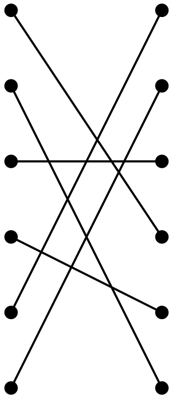
One can test if τ is a pattern of σ in time $f(\tau) \cdot |\sigma|$.

Twin-width

Twin-width of $(X, <_1, <_2)$:

- iteratively merge elements of X
- error between $A, B \subset X$ if they interleave for either $<_1$ or $<_2$
- minimize the error degree



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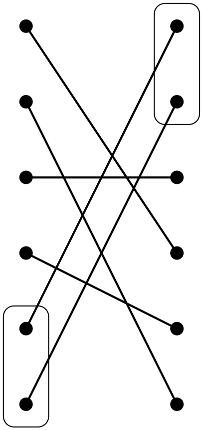

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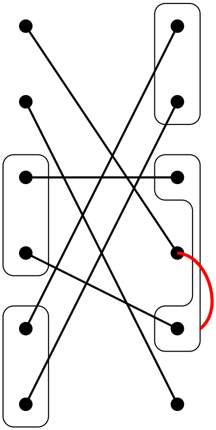

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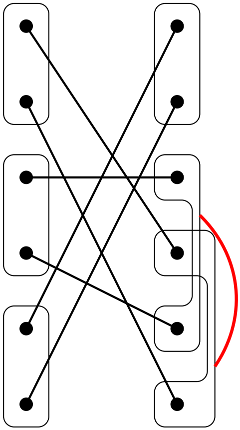

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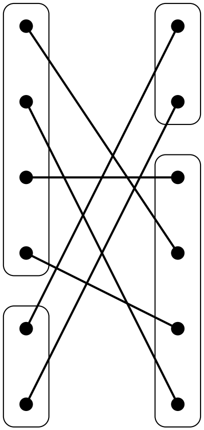

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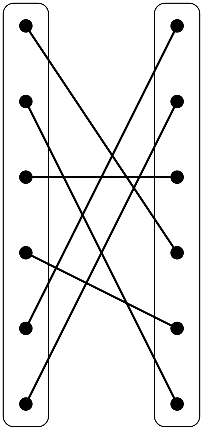

Guillemot–Marx Algorithm

Theorem (Guillemot–Marx '14)

One can test if τ *is a pattern of* σ *in time* $f(\tau) \cdot |\sigma|$.

Win–win argument:

Lemma

A class $\mathcal{C}$ *avoids a pattern if and only if it has bounded twin-width.*

Lemma

One can test if τ *is a pattern of* σ *in time* $f(\tau, \mathrm{tww}(\sigma)) \cdot |\sigma|$.

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We give a 'decomposition':

Theorem (BBGT)

For any pattern τ, there is a constant k such that any $\sigma \in \mathcal{F}(\tau)$ factorises as $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$, with σ_i separable.

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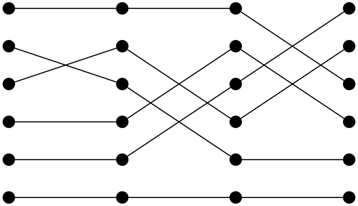

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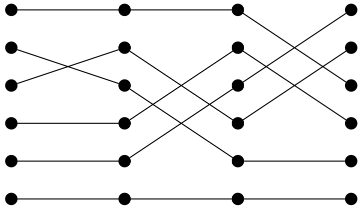
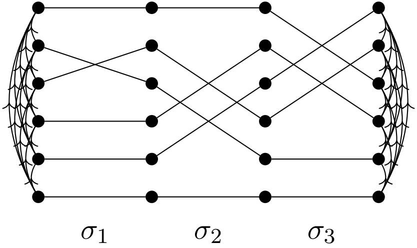


Fact

For any σ_1, σ_2, $\mathrm{tww}(\sigma_1 \circ \sigma_2) \leq f(\mathrm{tww}(\sigma_1), \mathrm{tww}(\sigma_2))$.

Transducing classes of bounded twin-width

Path representation of the factorisation:



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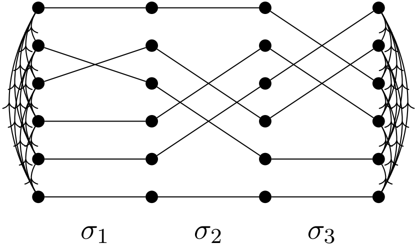

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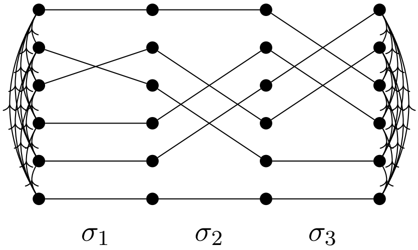

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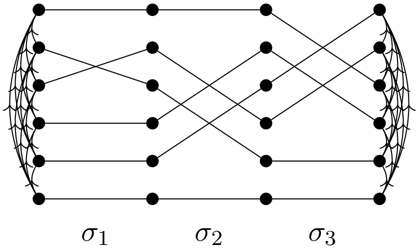

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path representation $\longrightarrow$ permutation $\longrightarrow$ graph

Proof overview (from very far away)

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Theorem (Pilipczuk, Sokołowski, Bourneuf, Thomassé, '23)

Classes of graphs with bounded twin-width are polynomially χ-bounded.

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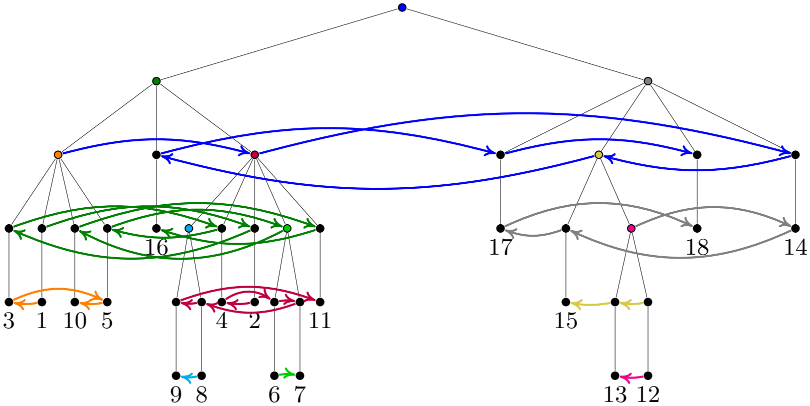
For permutations, this decomposition can be expressed with direct and skew sums, and a bounded number of products.

The Decomposition



Open Questions

- What is the smallest c such that $\mathrm{tww} = c$ transduces all of bounded twin-width (conjecture: $c = 4$).
- Applications of this factorisation?
- Computing shortest factorisations into separable permutations? (is it FPT? approximation?)
- Generalisation to matrices?