Twin-Width of Groups

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1st Workshop on Twin-width
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Open problems on twin-width

- Efficient approximation of twin-width

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Goal: anything interesting about twin-width and bounded degree

Strict twin-width

Definition

Strict twin-width $\mathrm{stww}(G)$: like twin-width, but

- natural contractions, without red edges
- bound the degree of the graphs in the sequence

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$$\max(\mathrm{tww}(G), \Delta(G)) \leq \mathrm{stww}(G) \leq \mathrm{tww}(G) + \Delta(G)$$

Strict twin-width is monotone under taking subgraphs

Powers of graphs

Power graph: $G^{(k)} = (V(G), \{xy \mid d_G(x,y) \leq k\})$

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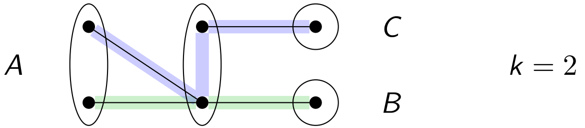
For $\mathcal{P}$ partition of $V(G)$, $G^k/\mathcal{P} \subseteq (G/\mathcal{P})^k$.



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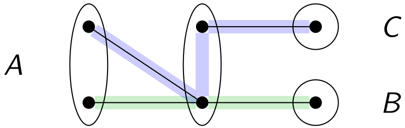

$k = 2$

$$G/\mathcal{P}_n, \ldots, G/\mathcal{P}_1 \text{ contraction sequence with degree } \leq d$$
$$\Rightarrow \quad G^k/\mathcal{P}_n, \ldots, G^k/\mathcal{P}_1 \text{ contraction sequence with degree } \leq d^k$$

Cayley graphs

For Γ group, S finite generating set, $\mathsf{Cay}(\Gamma, S)$ is:

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For any finite generating S_1, S_2, $\exists k$,

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All Cayley graphs of Γ have finite twin-width, or none do.

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3. Are groups useful for twin-width of graphs?

Groups and smallness

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Classes of graphs with bounded twin-width are small:
at most $c^n n!$ labelled graphs on n vertices.

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The class induced by any fixed Cayley graph is small.

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Groups with infinite twin-width

Theorem (Osajda)

Let $(G_n)_{n \in N}$ *be a sequence of graphs with*

- *bounded degree,*
- *bounded* $\mathrm{diam}(G_n)/\,\mathrm{girth}(G_n)$ *ratio,*
- *and* $\mathrm{girth}(G_{n+1}) \geq \mathrm{girth}(G_n) + 6$.

Then $(G_n)_{n \in N}$ *embeds isometrically into some Cayley graph.*

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Claim

The class of graphs with degree ≤ 6 and $\mathrm{diam} / \mathrm{girth} \leq 12$ is not small.

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Claim

The class of graphs with degree ≤ 6 and $\mathrm{diam} / \mathrm{girth} \leq 12$ is not small.

- Pick G bipartite cubic randomly.
- With constant probability, G has few cycles of length $\leq \log(n)/4$.
- Edit $O(n^{7/8})$ edges to ensure $\mathrm{girth}(G) \geq \log(n)/4$ and $\mathrm{girth}(G) \leq 3\log(n)$.

Group presentation

Classical presentation (of groups):

- S a set of 'generators'
- R a set of words on $S \cup S^{-1}$

$\langle S; R \rangle$ is the group generated by S, such that $\forall r \in R$, $r = 1$.

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Understanding $\langle S; R \rangle$ is hard:
given S, R, testing $\langle S; R \rangle = \{1\}$ is undecidable.

Small cancellation

$\langle S; G\rangle$ a graphical presentation (G is S-labelled)

$C'(\lambda)$ small cancellation condition:
if $p_1, p_2 \in G$ are distinct paths with the same labelling, then
if p_1 is contained in a cycle C,

$$|p_1| \leq |C|/\lambda$$

(and idem with p_2).

Lemma

If $\langle S; G\rangle$ satisfies $C'(6)$, for any word $w = w_1 \dots w_n$ over $S \cup S^{-1}$ such that $w = 1$, one can shorten w by using some equality given by some cycle of G.

Proof based on Euler formula.

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Theorem

For any group Γ and generating set S, the following are equivalent:

- Γ *has finite twin-width,*
- Γ *admits an order $<$ s.t. $\forall x \in \Gamma$, $\mathrm{tww}(M_<(x)) < \infty$.*
- Γ *admits an order $<$ s.t. $\forall x \in S$, $\mathrm{tww}(M_<(x)) < \infty$.*

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Orderable groups have finite twin-width.

Uniform twin-width

Definition

The uniform twin-width of Γ is

$$\mathrm{utww}(\Gamma) = \min_{< \text{ total order}} \sup_{x \in \Gamma} \mathrm{tww}(M_<(x))$$

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Lemma

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For any group G, subgroup $H \subset G$,

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And many other lemmas of this form...
These give many groups of finite uniform twin-width.

Summary and questions

Results:

- Twin-width generalises to groups.
- There is a group with infinite twin-width.
- It gives a small class with unbounded twin-width.
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Results:

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- Many groups have finite twin-width, and it is preserved by several usual constructions.

Questions:

- Explicit construction for a group with infinite twin-width?
- Applications of twin-width to groups?
- Separate finite twin-width and finite uniform twin-width? Candidate: permutations on $\mathbb{Z}$.
- Any group with uniform twin-width other than 2 or ∞?